A SIMPLE FORMULA FOR EMITTANCE GROWTH DUE TO SPHERICAL ABERRATION IN A SOLENOID LENS

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Abstract

We analyse the beam dynamics in a solenoid without the paraxial approximation, including up to the fifth order term in the radial displacement. We use this analysis to derive expressions for the coefficients of spherical aberration in terms of the on-axis field profile of the solenoid. Under the thin lens approximation, a simple formula is derived for the growth of rms emittance resulting due to spherical aberration in a solenoid.

INTRODUCTION

A solenoid is often used to focus charged particle beams in the low energy section of accelerators [1,2]. The focal length of a solenoid varies as a function of the radial displacement of the incident particle from its axis. This gives rise to spherical aberration resulting in the blurring of image, like in the case of optical lenses. However, more importantly in terms of accelerator physics, it gives rise to growth in the rms emittance. This issue is of practical interest since the typical beam size in the low energy section of an ion accelerator is large and may be comparable to the bore radius of solenoid. The emittance growth resulting due to spherical aberration may have detrimental effects and should therefore be reduced to within the specified acceptable limit. In order to study the spherical aberration, we study the beam dynamics in a solenoid keeping the nonlinear terms in the radial displacement. Under the thin lens approximation, the solenoid only imparts a kick to the radial velocity and does not per perturb the radial co-ordinate significantly. This assumption makes the calculation of emittance growth easier and gives us a simple formula for the growth of rms emittance due to spherical aberration in a solenoid lens. To the best of our knowledge, such a formula is not widely available in literature.

BEAM DYNAMICS WITHOUT THE PARAXIAL APPROXIMATION

We start with the following general expression for the components of axisymmetric magnetic field of a solenoid [3]

$$B_z(r,z) = B(z) - \frac{r^2}{4}B''(z) + \frac{r^4}{64}B''''(z) - \quad \cdots, \qquad (1)$$

$$B_r(r,z) = -\frac{r}{2}B'(z) + \frac{r^3}{16}B'''(z) - \quad \cdots, \qquad (2)$$

where z is the distance along the solenoid axis, r is the radial distance from the solenoid axis, and the prime denotes a derivative with respect to z. We are using the notation $B(z) = B_z(0,z)$. Note that $B_\theta(r,z) = 0$ due to axisymmetric nature of the field. Equation of motion along the radial direction is given by

$$\gamma m\ddot{r} = \gamma m r\dot{\theta}^2 + qr\dot{\theta}B_z, \qquad (3)$$

where the dot denotes the time derivative, $\gamma = \sqrt{1 - v^2/c^2}$ is the usual Lorentz factor, c is the speed of light in vacuum, v is the speed, q is the charge and m is mass of the charged particle. Motion along the azimuthal direction is described using Busch's theorem [3], which gives $\gamma m r^2\dot{\theta} = -q\psi/2\pi,$ where ψ is the magnetic flux enclosed by circle of radius r given by the radial distance r of the particle from the axis, passing through the particle and centred on the solenoid axis. Note that it is assumed that initially, $\dot{\theta} = 0$ and the particle is in field-free region. In the paraxial approximation, we take $\psi = \pi r^2 B(z)$. However, for the more general case, one has to take the variation of magnetic field over the surface area and doing this, we get

$$\dot{\theta} = -\frac{qB(z)}{2\gamma m} + \frac{qB''(z)}{16\gamma m}r^2 - \frac{qB''''(z)}{384\gamma m}r^4 + \cdots \qquad (4)$$

Putting Eq. 4 in Eq. 3 and keeping terms up to fifth order in r, we get the following equation for evolution of the radial co-ordinate

$$\ddot{r} = -\frac{q^2B^2(z)}{4\gamma^2m^2}\left[r - \frac{B''(z)}{2B(z)}r^3 + \left[\frac{B''''(z)}{32B(z)} + \frac{3\{B''(z)\}^2}{64B^2(z)}\right]r^5\right]. \qquad (5)$$

Next, we invoke the thin lens approximation. As described in Ref. 2, the condition for thin lens approximation is that $qBL/2\gamma mv_z << 1$, where B and L are the magnetic field and length of the equivalent hard edge solenoid lens, and v_z is the longitudinal component of particle velocity. Under this approximation, we assume that the radial co-ordinate does not change significantly inside the solenoid, but the slope $r'=dr/dz$ gets a net impulse. Also, we change the independent variable from t to z and assume that v_z remains constant in the solenoid. Integrating Eq. 5 under these conditions, we get the following expression for the kick $\Delta r'$

$$\Delta r' = -\frac{r}{f_0}\left[1 + C_1r^2 + C_2r^4\right], \qquad (6)$$

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where the focal length f_0 of the thin solenoid lens under paraxial approximation is given by

$$\frac{1}{f_0} = \left(\frac{q}{2\gamma m v_z}\right)^2 \int_{-\infty}^{+\infty} B^2(z)dz, \qquad (7)$$

and the spherical aberration coefficients C_1 and C_2 are given by

$$C_1 = \frac{1}{2}\frac{\int_{-\infty}^{+\infty}\{B'(z)\}^2 dz}{\int_{-\infty}^{+\infty} B^2(z)dz},\ C_2 = \frac{5}{64}\frac{\int_{-\infty}^{+\infty}\{B''(z)\}^2 dz}{\int_{-\infty}^{+\infty} B^2(z)dz}. \qquad (8)$$

The physical meaning of C_1 and C_2 is that for an off-axis particle incident at a distance r from the solenoid axis, the fractional reduction in the focal length is given by $C_1 r^2$ and $C_2 r^4$ due to third and fifth order spherical aberration respectively. Expression for C_1 is same as derived earlier by Sarma et al [4]. Note that C_1 and C_2 will always be positive, as required by Scherzer's theorem [5].

FORMULA FOR EMITTANCE GROWTH

The rms emittance ε_r of a beam for the case where the particles have only radial component of transverse velocity, is given by

$$\varepsilon_r^2 = \langle r^2\rangle\langle r'^2\rangle - \langle rr'\rangle^2, \qquad (9)$$

and if such a beam has azimuthal symmetry, it can be shown that $\varepsilon_x = \varepsilon_y = \varepsilon_r/2$. For the calculation of emittance growth, we assume a cold beam at the entrance of the solenoid and therefore r' = 0 for all the particles before entering the solenoid. At the exit of the solenoid, in the field free region, $\dot{\theta}$ =0, as it is obvious from Eq. 4. In the thin lens approximation, we assume that r is unchanged. Putting the expression for r' in Eq. 9, we get

$$\varepsilon_r^2 = \frac{C_1^2}{f_0^2}\left[\langle r^2\rangle\langle r^6\rangle - \langle r^4\rangle^2\right] + \frac{2C_1C_2}{f_0^2}\left[\langle r^2\rangle\langle r^8\rangle - \langle r^4\rangle\langle r^6\rangle\right]$$

$$+\frac{C_2^2}{f_0^2}\left[\langle r^2\rangle\langle r^{10}\rangle - \langle r^6\rangle^2\right]. \qquad (10)$$

Assuming uniform distribution for the incident beam, it can be shown that $\langle r^n\rangle = 2R^n/(n+2)$, where R is the hard edge radius of the beam. Putting this result in Eq. 10, we get the following expression for the emittance growth due to spherical aberration in a thin solenoid lens

$$\varepsilon_{x,y} = \frac{R^4}{2\sqrt{6}f_0}\sqrt{\frac{C_1^2}{12} + \frac{C_1C_2}{5}R^2 + \frac{C_2^2}{8}R^4}. \qquad (11)$$

The above expression can be used for estimating the upper limit on the coefficients C_1 and C_2 for a given value of allowable emittance growth. For example, in the Low Energy Beam Transport line for the H^- front-end linac under development at RRCAT, two solenoids will be used with hard edge length of 300 mm, bore radius 80 mm and field strength of 3.5 kG [6]. For 50 keV beam energy, we calculate f_0 = 10.7 cm. The average beam radius in the solenoid is 27 mm. The unnormalised rms beam emittance from the ion source is approximately 20 mm-mrad and it will be desirable that the emittance does not increase by more than 10% of its value in the solenoid. For calculating the upper limit on C_1, we assume that C_2 = 0, which gives $\varepsilon_{x,y} = C_1R^4/12\sqrt{2}f_0$. Putting the numbers in this formula, we obtain $C_1 < 6.8$ per m^2. Using Eq. 6, one can find that the fractional change in the focal length over the beam radius is around 0.5% in this case. Using an approximate formula given in Ref. 7, $C_1 = 1/(3.24\, bl)$, where b is the bore radius of the solenoid and l is the length of the solenoid, one obtains, that C_1 = 12.9 per m^2 can be achieved. Some special techniques may be needed to improve C_1 to the desired value. We can also use Eq. 11 to estimate the upper limit for C_2, which is obtained as 7.65×10^3 per m^4, under the conditions mentioned above and assuming C_1= 0.

It is also possible to extend our analysis to include the space charge. Further studies on comparison of our analytic calculation with numerical simulation will be taken up later.

In conclusion, we have performed an analysis of beam dynamics in a thin solenoid lens without the paraxial approximation and used it to estimate the emittance growth due to spherical aberration present in the solenoid. Our analysis can be used to estimate the limiting value of spherical aberration that can be tolerated for a given limiting value of acceptable emittance growth.

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REFERENCES

[1] A. Chao and M. Tigner, *Handbook of Accelerator Physics and Engineering* (World Scientific, Singapore) (1998).

[2] V. Kumar, Am. J. Phys. 77 (2009) 737.

[3] M. Reiser, *Theory and Design of Charged Particle Beams* (John Willey & Sons, New York) (1994).

[4] P. R. Sarma, S. K. Pattanayak and R. K. Bhandari, Nuclear Instrum. and Meth. Phys. Res. A 426 (2000) 243.

[5] O. Scherzer, Z. fur Phys. 101 (1936) 593.

[6] C. B. Patidar and P. Shrivastava, RRCAT Report, RRCAT 2010-07 (2010).

[7] A. V. Burov, FERMILAB-TM-2117 (2000).